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ON THE STRUCTURE OF GENERALIZED SOLUTIONS OF THE ONE-DIMENSIONAL EQUATIONS OF A POLYTROPIC VISCOUS GAS*

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A model system of equations that defines the unsteady one-dimensional flow of a visoucs gas is considered on the assumption that the pressure is determined by the adiabatic Poisson law. Generalized solutions are investigated in the class of discontinuous functions, a class of correctness is separated, and the structure of solutions of this class is clarified. It is shown that the initial velocity discontinuities are instantly smoothed out, and from the discontinuity points of the initial density, lines of contact discontinuity are formed. These lines exist for an infinite time, and the pressure jumps on them vanish exponentially.

1. Definition of the generalized solution. The initial boundary value problem (problem A) is investigated

$$\rho\,(u_t + uu_x) = \mu u_{xx} - p_x,\ \rho_t + (\rho u)_x = 0,\ p = \rho^\gamma \tag{1.1}$$
$$u\,(0,\ t) = u\,(1,t) = 0,\ u\,(x,\ 0) = u_0\,(x),\ \rho\,(x,\ 0) = \rho_0\,(x);$$
$$\mu > 0,\ \gamma \geqslant 1$$

in region Q of two variables x, t, $0 < x < 1$, $0 < t < T$. Here u is the velocity, ρ is the density, and p is the pressure. If the initial data are fairly smooth and satisfy the consistency conditions, this problem is uniquely solvable /1/. Let us investigate the case where the initial data are discontinuous

Definition 1. The measurable functions u, ρ will be called the generalized solution (GS) of problem A, when $u \in L_4\,(Q)$, $\operatorname{vrai\,min} \rho\,(Q) > 0$, $\operatorname{vrai\,max} \rho\,(Q) < \infty$, and the first two equations of (1.1) are satisfied in the sense of the integral identities

$$\int_0^T \int_0^1 (\rho u \varphi_t + \rho u^2 \varphi_x + p\varphi_x + \mu u \varphi_{xx})\,dx\,dt + \int_0^1 \rho_0 u_0 \varphi\,(x, 0)\,dx = 0$$
$$\int_0^T \int_0^1 (\rho\psi_t + \rho u \psi_x)\,dx\,dt + \int_0^1 \rho_0 \psi\,(x, 0)\,dx = 0$$

for arbitrary functions $\varphi, \psi \in W_2^1\,(Q) \cap L_\infty\,(0, T; L_2\,(0, 1))$ with the conditions

$$\varphi_{xx} \in L_2\,(Q),\ \varphi\,(x, T) = \psi\,(x, T) = 0, \quad \varphi\,(0, t) = \varphi\,(1, t) = 0$$
$$\int_0^1 \rho\,(x, t)\,dx = \int_0^1 \rho_0\,(x)\,dx \equiv \alpha, \quad 0 \leqslant t \leqslant T$$

We take into account here that the left side of the first equation of (1.1) may be represented, by virtue of the second equation, in the form $(\rho u)_t + (\rho u^2)_x$.

Subsequently we shall consider only stable GS.

Definition 2. The GS of problem A will be called stable, if it is the limit of classic solutions (CS).

A more exact definition will be given later.

To clarify the structure of stable GS we will use its other definition (a proof of the equivalence of the definitions will be given later).

First let us consider one of the properties of CS.

The CS of problem A can be obtained from the CS of the following problem (problem B) /1/:

$$u_t' = \mu\,(\rho' u_\xi')_\xi - p_\xi', \quad v_t' = u_\xi', \quad v = \rho^{-1} \tag{1.2}$$
$$\xi,\ t \in Q':\ 0 < \xi < \alpha,\ 0 < t < T$$

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$$u'(0,t)=u'(\alpha,t)=0,\ u'(x,0)=u_0'(x),\ \rho'(x,0)=\rho_0'(x)$$

where ξ, t are the Lagrangian mass coordinates. The change of variables $\xi, t \to x, t$

$$x(\xi,t)=\int_0^\xi v'(\eta,t)\,d\eta \tag{1.3}$$

transfers the CS of problem B into a CS of problem A. The connection of initial data of the two problems is generated by the same change of variables. For example for the function u we have

$$u_0'(\xi)=u_0\Big(\int_0^\xi v_0'(\eta)\,d\eta\Big),\quad u_0(x)=u_0'\Big(\int_0^x \rho_0(y)\,dy\Big)$$

We deal similarly when defining the stable GS of problem A. We determine, first, the GS of problem B and then, making the change of variables (1.3), call it the stable GS of problem A.

We introduce the following notation:

$$\Omega'=\{\xi: 0<\xi<\alpha\},\ Q_\delta'=Q'\cap(\delta<t<T)$$
$$|u'|_\delta=\operatorname*{vraimax}_{\delta\leqslant t\leqslant T}\|u'(\xi,t)\|+\|u_\xi'\|_\delta$$

where $\|u'\|, \|u'\|_\delta$ are the norms in $L_2(\Omega'), L_2(Q_\delta')$, respectively, and $\|u'\|_V$ is the norm of $u'(\xi)$ in the space $V[\Omega']$ of functions of limited variation:

Let us rewrite system (1.2) in the form

$$u_t' \times \mu(\ln v')_{\xi t}-p_\xi',\ v_\xi'=u_\xi' \tag{1.4}$$

Definition 3. We call the measurable functions u', ρ' the GS of problem B, if $u'\in L_4(Q')$, $\operatorname{vraimin}\rho'(Q')>0$, $\operatorname{vraimax}\rho'(Q')<\infty$ and (1.4) are satisfied in the sense of integral identities

$$\iint_{Q'}(\mu\ln v'\varphi_{\xi t}+u'\varphi_t+p'\varphi_\xi)\,d\xi\,dt+\int_{\Omega'}(\mu\ln v_0'\varphi_\xi(\xi,0)+u_0'\varphi(\xi,0))\,d\xi=0$$
$$\iint_{Q'}(u'\psi_\xi-v'\psi_t)\,d\xi\,dt-\int_{\Omega'}v_0'\psi(\xi,0)\,d\xi=0,\quad v=\rho^{-1}$$

for arbitrary functions $\varphi(\xi,t), \psi(\xi,t)\in W_2^1(Q')\cap L_\infty(0,T;L_2(0'))$ with the conditions

$$\varphi\in L_\infty(0,T,W_2^1(\Omega')),\ \varphi_{\xi t}\in L_2(Q'),\ \varphi(\xi,T)=\psi(\xi,T)=0$$
$$\varphi(0,t)=\varphi(\alpha,t)=0,\quad \int_{\Omega'}v'(\xi,t)\,d\xi=1,\quad 0<t<T$$

Definition 4. We call the functions $u(x,t), \rho(x,t)$ the stable GS of problem *A*, if they were obtained from some GS of problem B by the change of variables (1.3).

2. The existence and uniqueness. *Theorem* 1. The GS of problem B is unique.

Proof. Assume that two different GS exist. Then for their difference $u'=u_1'-u_2', v'=v_1'-v_2'$ by virtue of Definition 3, the equation

$$\iint_{Q'}(v'L_1(\varphi,\psi)+u'L_2(\varphi,\psi))\,d\xi\,dt=0 \tag{2.1}$$
$$L'\equiv\mu a\varphi_{\xi t}+b\varphi_\xi-\psi_t,\ L_2=\varphi_t+\psi_\xi$$
$$a=(\ln v_1'-\ln v_2')(v_1'-v_2')^{-1},\quad b=(p_1'-p_2')(v_1'-v_2')^{-1}$$

is satisfied. The assumptions on solutions imply that a, b are bounded measurable functions and $\operatorname{vrai\,min} a(Q')=a_0>0$. We approximate them in $L_2(Q')$ by fairly smooth functions $a_\varepsilon, b_\varepsilon$ such that their absolute values are uniformly bounded with respect to ε and $2\cdot\operatorname{vrai\,min} a_\varepsilon(Q')\geqslant a_0$.

We rewrite (2.1) in the form

$$\iint_{Q'}[\mu(a-a_\varepsilon)v'\varphi_{\xi t}+(b-b_\varepsilon)v'\varphi_\xi+L_1^\varepsilon(\varphi,\psi)v'+L_2(\varphi,\psi)u']\,d\xi\,dt=0$$

To obtain a contradiction it is sufficient to prove the solvability of the problem

$$L_1^\xi(\varphi,\psi)=f,\ L_2(\varphi,\psi)=g \tag{2.2}$$

for functions φ, ψ of the class indicated in Definition 3, where f, g are arbitrary functions from $C^{\circ\infty}(Q')$. It is furthermore necessary for each f, g to have an estimate

$$\| \varphi_{\xi t} \|_0 + \| \varphi_\xi \|_0 \leqslant c \tag{2.3}$$

that is uniform with respect to ε. Here and subsequently c is a positive constant.

We introduce the new variable $t \to T - t$, then (2.2) becomes equivalent to the following problem:

$$\psi_t = \mu a_\varepsilon \psi_{\xi\xi} - b_\varepsilon \int_0^t \psi_{\xi\xi}\, d\tau + F, \quad \psi(\xi, 0) = 0 \tag{2.4}$$

$$F = f + b_\varepsilon \int_0^t g_\xi\, d\tau - \mu a_\varepsilon g_\varepsilon, \quad \psi_\xi(0, t) = \psi_\xi(\alpha, t) = 0$$

The solvability of the linear problem (2.4) can be proved, for instance, by the Galerkin method.

The estimate of (2.3) follows from the formula

$$\varphi_\xi = \int_0^t \psi_{\xi\xi}\, d\tau - \int_0^t g_\xi\, d\tau$$

if the estimate of $\| \psi_{\xi\xi} \|_0$ is obtained, which is achieved by multiplying (2.4) by $\psi_{\xi\xi}$. The theorem is proved.

Theorem 2. If $u_0' \in L_2(\Omega')$, $v_0' \in V(\Omega')$, $\operatorname{vraimax} v_0'(\Omega') \leqslant \varkappa$, $\varkappa \cdot \operatorname{vrai\,min} v_0'(\Omega') \geqslant 1$, $\varkappa = \text{const} > 0$, then problem B has a GS.

Proof. Let the functions $u_{0\varepsilon}', v_{0\varepsilon}'$ be fairly smooth and suppose it can be the initial data for the CS of problem B. We shall use them to approximate the initial data u_0', v_0' in the following sense:

$$\| u_0' - u_{0\varepsilon}' \| + \| v_0' - v_{0\varepsilon}' \|_V \to 0, \quad \int_{\Omega'} v_{0\varepsilon}'\, d\xi = 1, \quad \varepsilon \to 0$$

Note that this approximation, which satisfies the consistency condition, and always be made, since $u_{0\varepsilon}'$ must converge only in $L_2(\Omega')$.

Let us consider problem B with initial data $u_{0\varepsilon}', v_{0\varepsilon}'$. For its solution according to /1/ $u_\varepsilon', v_\varepsilon'$ has the following estimates uniform with respect to ε:

$$| u_\varepsilon' |_0 + \| v_{\varepsilon t}' \|_0 \leqslant c, \quad c^{-1} \leqslant v_\varepsilon' \leqslant c \tag{2.5}$$

where c depends on $\| u_0' \|, \varkappa$. In particular, the u_ε' are by the imbedding theorem, uniformly bounded in $L_4(Q')$ with respect to ε.

Using the notation

$$\omega_\varepsilon = \ln v_\varepsilon'(\xi_2, t) - \ln v_\varepsilon'(\xi_1, t)$$

We integrate the first of equation (1.2)

$$\omega_\varepsilon + \int_0^t q\omega_\varepsilon\, d\tau = G, \quad q = -\mu\, [p'(\xi_2, t) - p'(\xi_1, t)]\, \omega_\varepsilon^{-1}$$

$$G = \frac{1}{\mu} \int_{\xi_1}^{\xi_2} (u_\varepsilon' - u_{0\varepsilon}')\, d\eta + \ln v_{0\varepsilon}'(\xi_2) - \ln v_{0\varepsilon}'(\xi_1)$$

from which by Gronwall's lemma we have

$$\sup_{0 \leqslant t \leqslant T} \| v_\varepsilon'(t) \|_V \leqslant c \tag{2.6}$$

By the Helly theorem /2/ we can select from the family of function v_ε' that satisfy conditions (2.5) and (2.6) a sequence which converges almost everywhere in Q'. The set u_ε' is weakly compact in $L_2(Q')$. Thus some sequence of CS $(u_\varepsilon', v_\varepsilon')$ of problem B exists which, when $\varepsilon \to 0$, converges to measurable functions u', v' in the following sense: $u_\varepsilon' \to u'$ weakly converges in $L_2(Q')$ and $v_\varepsilon' \to v'$ converges almost everywhere in Q'.

Obviously (u', v') is the GS of problem B. Moreover, it follows from (2.5) that the derivatives u_ξ', v_t' exist, and

$$| u' |_0 \leqslant c, \; \| u' \|_{L_4(Q')} \leqslant c, \; c^{-1} \leqslant v' \leqslant c, \; \| v_t' \|_0 \leqslant c \tag{2.7}$$

The theorem is proved.

Let (u', v') be the GS of problem B. Let us make the change of variables (1.3):

$$u(x, t) = u'(\xi(x, t), t), \; v(x, t) = v'(\xi(x, t), t) \tag{2.8}$$

Here $\xi(x, t)$ is the inverse function of $x(\xi, t)$ when t is fixed

$$x(\xi, t) = \int_0^{\xi} v'(\eta, t)\, d\eta, \quad x = \int_0^{\xi(x,t)} v'(\eta, t)\, d\eta \tag{2.9}$$

By Definition 4 the functions (u, v) are the stable GS of problem A. We shall show that they are stable GS in the meaning of Definition 2, i.e. satisfy the integral equations from Definition 1, and the bounds of CS of problem A. In passing, we shall give a more precise definition of this passage to the limit.

Let u_n', v_n' be that sequence of CS of problem B from the Theorem 2, which converges to the functions u', v'. We change on CS u_n', v_n' to Euler coordinates x, t, using formulae

$$x(\xi, t) = \int_0^{\xi} v_n'(\eta, t)\, d\eta, \quad \frac{\partial(x, t)}{\partial(\xi, t)} = v_n' \tag{2.10}$$

$$u_n(x, t) = u_n'(\xi_n(x, t), t), \quad v_n(x, t) = v_n'(\xi_n(x, t), t)$$

where $\xi_n(x, t)$ is a function that for fixed t is the inverse of $x(\xi, t)$, i.e.

$$x = \int_0^{\xi_n(x,t)} v_n'(\eta, t)\, d\eta$$

Obviously (u_n, v_n) is the CS of problem A, hence the integral equations from Definition 1 are satisfied by it, if we take the functions u_{0n}, v_{0n} as the initial data.

Lemma 1. The limit relation $\lim v_n(x, t) = v(x, t)$ holds for almost all $x, t \in Q$ as $n \to \infty$.

Proof. Suppose the following limit exists at the point ξ_0, t_0:

$$\lim_{n\to\infty} v_n'(\xi_0, t_0) = v'(\xi_0, t_0) \tag{2.11}$$

We shall show that at the corresponding point (x_0, t_0)

$$x_0 = \int_0^{\xi_0} v'(\eta, t_0)\, d\eta$$

the lemma is satisfied.

We have

$$x_0 = \int_0^{\xi_0} v'(\eta, t_0)\, d\eta = \int_0^{\xi_{0n}} v_n'(\eta, t_0)\, d\eta, \quad \xi_{0n} = \xi_n(x_0, t_0)$$

Hence

$$\int_{\xi_0}^{\xi_{0n}} v_n'(\eta, t_0)\, d\eta = \int_0^{\xi_0} [v'(\eta, t_0) - v_n'(\eta, t_0)]\, d\eta$$

Hence the sequence $\xi_n(x_0, t_0)$ has the limit

$$\lim_{n\to\infty} \xi_n(x_0, t_0) = \xi_0$$

Furthermore

$$|v_n(x_0, t_0) - v(x_0, t_0)| = |v_n'(\xi_{0n}, t_0) - v'(\xi_0, t_0)| \leqslant$$
$$|v_n'(\xi_{0n}, t_0) - v_n'(\xi_0, t_0)| + |v_n'(\xi_0, t_0) - v'(\xi_0, t_0)|$$

By the same token the lemma for point x_0, t_0 is proved owing to the uniform continuity of the function v_n'. The validity of the lemma for almost all $x, t \in Q$ follows from (2.11) being satisfied almost everywhere, and the Jacobian of the change of variables (2.10) is uniformly bounded above and below.

Lemma 2. The sequence $u_n(x, t)$ weakly converges in $L_2(Q)$ to $u(x, t)$.

Proof. By virtue of inequality (2.5)

$$|u_n|_0 \leqslant c, \; \|u_n\|_{L_2(Q)} \leqslant c \tag{2.12}$$

where the second estimate is the corollary of the first. Note that these estimates depend only on $\|u_0\|$, vraimax $v_0(\Omega)$, vraimin $v_0(\Omega)$.

Let $u_*(x, t)$ be the limit of some subsequence $\{u_k\} \subset \{u_n\}$ which converges weakly in $L_2(Q)$. Then for any function $\varphi(x, t) \in \overset{\circ}{C}{}^{\infty}(Q)$

$$\lim_{k\to\infty} \iint_Q u_k \varphi\, dx\, dt = \iint_Q u_* \varphi\, dx\, dt = \iint_{Q'} u_*'(\xi, t)\, \varphi'(\xi, t)\, v'(\xi, t)\, d\xi\, dt$$

$$\left(\varphi'(\xi, t) = \varphi(x(\xi, t), t),\; x(\xi, t) = \int_0^{\xi} v'(\eta, t)\, d\eta\right)$$

On the other hand

$$\iint_Q u_k \varphi\, dx\, dt = \iint_{Q'} u_k (x_k(\xi, t), t)\, \varphi (x_k(\xi, t), t)\, v_k'(\xi, t)\, d\xi \equiv \iint_{Q'} u_k'(\xi, t)\, \varphi_k'(\xi, t)\, v_k'(\xi, t)\, d\xi\, dt$$

$$\left(x_k(\xi, t) = \int_0^\xi v_k'(\eta, t)\, d\eta\right)$$

Since $\lim x_k(\xi, t) = x(\xi, t)$ everywhere in Q', and $u'_k \to u'$ weakly converges in $L_2(Q')$ hence

$$\iint_Q u_* \varphi\, dx\, dt = \iint_Q u \varphi\, dx\, dt$$

Consequently $u_* = u$ almost everywhere in Q.

Since the converging sequence u_k is arbitrary, the whole sequence u_n is weakly convergent in $L_2(Q)$ to u.

Lemma 3. The sequence u_n converges in $L_2(Q)$ to u.

Proof. In view of Lemma 2, it is sufficient to prove the compactness u_n in $L_2(Q)$. The compactness follows from estimate (2.12) and the estimate

$$\int_0^{T-\delta} \int_0^1 |u_n(x, t+\delta) - u_n(x, t)|^2\, dx\, dt \leqslant c\delta^{1/2}$$

Because of the uniform boundedness below and above of the Jacobian v_n', of the change of variables $\xi, t \to x, t$ on the solution u_n', v_n' this estimate is equivalent to

$$\int_0^{T-\delta} \int_0^\alpha |u_n'(\xi, t+\delta) - u_n'(\xi, t)|^2\, d\xi\, dt \leqslant c\delta^{1/2}$$

Let us obtain it . Let $w_n = u_n'(\xi, t+\delta) - u_n'(t)$. We integrate the first equation of system (1.2) with respect to t from t to $t+\delta$ and multiply it by w_n. We obtain

$$w_n^2 = \frac{\partial}{\partial \xi}\left(w_n \int_t^{t+\delta} \sigma_n\, d\tau\right) - w_{n\xi} \int_t^{t+\delta} \sigma_n\, d\tau, \quad \sigma_n = \mu \rho_n' u_{n\xi}' - p_n'$$

Integrating this equation and applying the estimates (2.5), we obtain the required inequality.

Lemmas 1 and 3, and estimates (2.12) enable us to pass to limit in the integral equations of Definition 1 written for u_n, v_n.

Thus the $u(x, t), v(x, t)$ is a stable solution of the GS of problem A in the sense of Definition 2. Theorem 1 implies that there is no other GS of this problem.

Let us summarize the above.

Theorem 3. Let $u_0(x) \in L_2(\Omega)$, $v_0(x) \in V(\Omega)$, $\text{vrai max}\ v_0(\Omega) < \infty$, $\text{vrai min}\ v_0(\Omega) > 0$. Then problem A is uniquely solvable in the class of stable GS.

3. The structure of stable GS.

To clarify the differential properties of a stable GS of problem A it is convenient to use Definition 4. First, we investigate the properties of the functions $u'(\xi, t), v'(\xi, t)$.

Lemma 4. Let u_δ', v_δ' be the GS of problem B in the subregion $Q_\delta' \subset Q'$ with initial data

$$u_\delta'(\xi, \delta) = u'(\xi, \delta), \quad v_\delta'(\xi, \delta) = v'(\xi, \delta)$$

Then, almost everywhere in Q_δ' we have $u' = u_\delta'$, $v' = v_\delta'$.

Proof. Instead of satisfying integral relation we may stipulate in Definition 3, the satisfaction of the integral relations:

$$\int_{t_1}^{t_2} \int_0^\alpha (\mu \ln v' \varphi_{\xi t} + u' \varphi_t + p' \varphi_\xi)\, d\xi\, dt = I_1(t_2) - I_1(t_1)$$

$$\int_{t_1}^{t_2} \int_0^\alpha (u' \psi_\xi - v' \psi_t)\, d\xi\, dt = -I_2(t_2) + I_2(t_1)$$

$$I_1 = \int_{\Omega'} (\mu \ln v' \varphi_\xi + u' \varphi)\, d\xi, \quad I_2 = \int_{\Omega'} v' \psi\, d\xi, \quad \lim_{t \to 0} I_i(t) = I_i(0)$$

for any functions φ, ψ that satisfy the same conditions in Definition 3, but without the condition that $\varphi = \psi = 0$ when $t = T$.

The proof of the equivalence of the integral identities is similar to that given in /3/. With this definition u', v' is the GS of problem B in region Q_δ' with initial data $u'(\xi, \delta), v'(\xi, \delta)$, almost everywhere in Q_δ' because of the uniqueness of $(u_\delta', v_\delta') = (u', v')$.

Lemma 5. A denumerable set $\{t_k\} \subset [0, T]$: $u_\xi'(\xi, t_k) \in L_2(\Omega')$, $v'(\xi, t_k) \in V(\Omega')$ exists which is everywhere dense.

Proof. Let u_n', v_n' be that sequence in Theorem 2 which converges to the GS of u', v'. Since $u_\xi' \in L_2(Q')$, then by Fubini's theorem a denumerable everywhere-dense set $\{t_k\} \subset [0, T]$ exists such that $u'(\xi, t_k) \in L_2(\Omega')$. By the estimate (2.6) we have $\|v_n'(t_k)\|_V \leqslant c$ for any t_k. Hence, it is possible according to Helly's theorem to separate the subsequence $\{v_m'\} \subset \{v_n'\}$ which converges for any t_k at every point $\xi \in \Omega'$. Thus we have $\|v'(t_k)\|_V \leqslant c$ for any t_k.

It can be shown similarly that

$$\|v'(t)\|_V \leqslant c, \ 0 \leqslant t \leqslant T \tag{3.1}$$

Theorem 4. In any subregion $Q_\delta' \subset Q'$ the function $u'(\xi, t)$ is according to Hölder uniformly continuous with exponents ½ and ¼ with respect to ξ and t, respectively.

Proof. By Lemma 5 we have for some t_k, $0 < t_k < \delta$

$$u'(\xi, t_k) \in W_2^1(\Omega'), \ v'(\xi, t_k) \in V(\Omega)'$$

Let u_k', v_k' be the GS of problem B in subregion $Q_{t_k}' \subset Q'$ with initial data $u'(\xi, t_k)$, $v'(\xi, t_k)$. We approximate these data by smooth functions u_{0h}', v_{0h}' in the following sense:

$$\|u_{0h}' - u'(\xi, t_k)\|_{W_2^1(\Omega')} + \|v_{0h}' - v'(\xi, t_k)\|_V \to 0, \ \int_0^\alpha v_{0h}' \, d\xi = 1, \ h \to 0$$

The CS u_h', v_h' of problem B converge to u_k', v_k' by virtue of uniqueness, and for it the estimate

$$|\sigma_h'|_{t_k} \leqslant c, \quad \sigma_h' = \mu\rho_h' u_{h\xi}' - p_h' \tag{3.2}$$

holds. It is obtained by multiplying the equation $u_t' = \sigma_\xi'$ by σ_ξ', using the identity

$$u_{t\xi}' = \frac{1}{\mu}\frac{\partial}{\partial t}(v'\sigma') + \frac{1}{\mu}\left(p' + v'\frac{\partial p'}{\partial v'}\right) v_t'$$

From (3.2) we have the estimates

$$\sup_{t_k \leqslant t \leqslant T} \|u_h'(t)\|_{W_2^1(\Omega')} \leqslant c, \quad \|u_{ht}'\|_{t_k} \leqslant c \tag{3.3}$$

which are uniform with respect to h, and ensure the validity of the lemma for the function $u_h'(\xi, t)$. By Arzela's lemma we can separate from the family u_h' the subsequence which converges uniformly in Q_{t_k}' to u_k'. The theorem is thus proved also for the function u'.

Let us define the properties of the function $v'(\xi, t)$. The original function is $v_0'(\xi) \in V(\Omega')$, and consequently has not more than a denumerable set of points of discontinuity.

Theorem 5. Let β_k be the points of discontinuity of the function $v_0'(\xi)$ $(k = 1, 2, \ldots)$. Then the function $v'(\xi, t)$ is continuous everywhere in Q', except on the lines $\xi = \beta_k$, and its discontinuities decrease exponentially along these lines i.e.

$$\begin{gathered} c_1 \exp(-c_2 t) \leqslant V_k(t) \leqslant c_3 \exp(-c_4 t) \\ V_k(t) = |v'(\beta_k + 0, t) - v'(\beta_k - 0, t)| \end{gathered} \tag{3.4}$$

where the constants c_i are independent of time.

Proof. All the information contained in the theorem follows from the formula

$$\begin{gathered} p' = \frac{\mu}{\gamma}\frac{\partial}{\partial t}\ln\left(1 + \int_0^t E(\xi, \tau)\, d\tau\right), \quad t > 0 \\ E(\xi, t) = \frac{\gamma}{\mu} p_0'(\xi) \exp\left\{\frac{\gamma}{\mu}\left[\int_0^t\int_0^\alpha ((u')^2 + p'v')\, d\xi d\tau + B(\xi, t)\right]\right\} \\ B(\xi, t) = -\int_0^\alpha v'(\eta, t)\int_\eta^\xi u'(\zeta, t)\, d\zeta\, d\eta - \int_0^\alpha v_0'(\eta)\int_0^\eta u_0'(\zeta)\, d\zeta\, d\eta + \int_0^\xi u_0'(\eta)\, d\eta \end{gathered} \tag{3.5}$$

Suppose that the formula is valid. Then the function $p'(\xi, t)$ is continuous with respect to ξ at the point of continuity of the function $p_0'(\xi)$, since $B(\xi, t)$ is continuous with respect to ξ. The estimate (3.4) is also obtainable from (3.5)

To prove the validity of formula (3.5) it is sufficient to do so for CS of problem B, since $v_n' \to v'$ almost everywhere in Q', and $u_n' \to u'$ uniformly in any subregion $Q_\delta' \subset Q'$ and in $L_2(Q')$.

We integrate the equations

$$\sigma' = -\mu (\ln \rho')_t - p', \quad u_t' = \sigma_\xi'$$

and obtain

$$p' \exp\left(\frac{\gamma}{\mu}\int_0^t p'\, d\tau\right) = p_0' \exp\left(-\frac{\gamma}{\mu}\int_0^t \sigma'\, d\tau\right) \tag{3.6}$$

$$z(\xi, t) = z(\eta, t) + \int_\eta^\xi [u'(\zeta, t) - u_0'(\zeta)]\, d\zeta, \quad z(\xi, t) = \int_0^t \sigma'(\xi, \tau)\, d\tau$$

We put

$$U_0(\xi) = \int_0^\xi u_0'(\eta)\, d\eta, \quad \Phi(\xi, t) = z(\xi, t) + U_0(\xi)$$

where the function Φ satisfies the relations

$$z(\xi, t) = \Phi(\eta, t) + \int_\eta^\xi u'(\zeta, t)\, d\zeta - U_0(\xi)$$

$$(v'\Phi)_t = \mu\Phi_{\xi\xi} + (\Phi_\xi\Phi)_\xi - p'v' - \Phi_\xi^2$$

First we multiply the first equation by $v'(\eta, t)$ and, then, integrate them with respect to the variables η, ξ, respectively

$$z(\xi, t) = \int_0^\alpha v'\Phi\, d\eta + \int_0^\alpha v'(\eta, t)\int_\eta^\xi u'(\zeta, t)\, d\zeta - U_0(\xi)$$

$$\int_0^\alpha v'\Phi\, d\xi = -\int_0^t\int_0^\alpha ((u')^2 + p'v')\, d\xi\, d\tau + \int_0^\alpha v_0'(\xi)\, U_0(\xi)\, d\xi$$

where use was made of the relation

$$\int_0^\alpha v'(\xi, t)\, d\xi = 1, \quad 0 \leqslant t \leqslant T$$

Substituting the expression obtained for $z(\xi, t)$ into the first of relations (3.6), we obtain formula (3.5)

From formula (3.5) and the estimate that is uniform with respect to time $|u'|_0 \leqslant c$ we can obtain a similar estimate $c^{-1} \leqslant v' \leqslant c$ /4/.

Theorem 6. The GS u', v' of problem B has almost everywhere in Q' finite derivatives $u_t', u_\xi', u_{\xi\xi}', v_t', v_\xi'$ and satisfies (1.2) almost everywhere in Q'.

Proof. It follows from estimates (2.7), (3.2) and (3.3) that $u_\xi', u_t', v_t', \sigma_\xi' \in L_2(Q_\delta')$ for any subregion $Q_\delta' \Subset Q'$, and almost everywhere in Q_δ' the equations $u_t' = \sigma_\xi'$, $v_t' = u_\xi'$ hold.

Then, by virtue of (3.1) the function v' has a finite derivative v_ξ' almost everywhere in Q'. The existence of finite derivatives σ_ξ', u_ξ' ensures a similar property for $u_{\xi\xi}'$.

Let us now consider the properties of the functions u, v. They are determined by the change of variables (2.8) and (2.9).

Lemma 6. The function $\xi(x, t)$ defined by (2.8) and (2.9) is Lipschitz continuous with respect to x, and Hölder continuous with respect to t with exponent ½ uniformly in Q.

The proof follows from the estimates (2.7) for the function v'

Let us formulate the final properties of the functions u, v.

Theorem 7. A stable GS u, v of problem A has the following structure:

1) in any subregion $Q_\delta \subset Q$ the function u is continuous Hölder with exponents ½ and ¼ of x and t, respectively;

2) the function $v(x, t)$ has for any t a bounded variation with respect to x, from the points of discontinuity of the original function $v_0(x)$ which do not intersect and do not approach the discontinuity line of the function v with respect to the variable x, and these discontinuities vanish exponentially (formula (3.4));

3) the functions u, v have almost everywhere in Q finite derivatives that appear in system (1.1), and satisfy that system almost everywhere.

Proof. Properties 1 and 2 follow from Theorems 4 and 5 and Lemma 6.

Let us prove property 3. Let u_n, v_n be that sequence in (2.10) which converges to u, v. By virtue of estimates (2.12), (3.2) and (3.3) in any subregion $Q_\delta \subset Q$ the derivatives u_t, u_x v_t, $(\rho u)_x$, σ_x exist as functions of $L_2(Q_\delta)$, where $\sigma = \mu u_x - p$, and the equations

$$\rho\,(u_t + uu_x) = \sigma_x,\ \rho_t + (\rho u)_x = 0$$

hold (in $L_2(Q_\delta)$). Since the function $v(x,t)$ has for any t for almost all x a finite derivative v_x, then from the equation $\sigma = \mu u_x - p$ it follows that the finite derivate u_{xx} exists almost everywhere. The theorem is proved.

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LOCALIZATION OF GAS-DYNAMIC PROCESSES AND STRUCTURE WHEN THE MATERIAL IS COMPRESSED ADIABATICALLY, IN THE PEAKING MODE*

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Adiabatic compression of gas by a piston, the pressure on which increases in the peaking mode, is studied. The entropy is distributed over the mass. A class of selfsimilar solutions (the *LS* mode) is constructed and its properties are studied. It is shown that the effective dimensions of the compression wave decrease with time and all gas-dynamic perturbations are localized within a finite mass of the gas. The solutions obtained are characterized by the presence of a structure (inhomogeneities) in the density and temperature. The compression occurs without the formation of shock waves.

The peaking mode, i.e. the precesses in which any quantities may become infinite in a finite period of time, have a number of unusual properties. Thus the development of the peaking modes in continua is accompanied by localization ("inertia") of the diffusion processes and the formation of non-stationary dissipative structures /1-3/.

Another example is offered by an isentropic (optimal) compression of a finite mass of gas to superhigh densities /2,4-7/**. Such a process takes place when the pressure acting on the compressing piston increases as follows (the *S* mode):

$$P\,(0,\ t) = P_0\,(t_f - t)^n,\quad n = -2\gamma\,(N+1)/(2 + (N+1)(\gamma - 1));$$
$$t_0 \leqslant t \leqslant t_f$$

where $N = 0,\ 1,\ 2$ is a geometrical index, γ is the adiabatic index and t_f denotes the instant of peaking.

The problem of the adiabatic compression of a cold gas initially at rest, by a piston acted upon by a pressure which varies with time according to a more general law, with peaking at any $n < 0$, is considered below for the case when $N = 0$.

Another generalization consists of the fact that the entropy of the gas depends on the Lagrangian mass coordinate $x \geqslant 0$ is such a manner, that $P\,(x,\ t) = a_0 x^\delta \rho^\gamma$ for all $t_0 \leqslant t < t_f$. Such a distribution of entropy in the medium arises e.g. behind the shock wave front moving through the gas, with velocity varying with time according to a power law.

Selfmodelling solutions are constructed for $n > -2\gamma/(\gamma + 1)$ (the *LS* mode) corresponding to

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** See also: Kazhdan Ya.M. On the problem of adiabatic compression of gas by a spherical piston. Preprint In-ta prikl. matem. Akad. Nauk SSSR, Moscow, No.89, 1975.

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